Research and Collaboration in Number Theory

Sabbatical report for Phil Moore

Goals and objectives

Because we utilize student collaboration extensively in our mathematics classes at LCC, I proposed in my sabbatical to explore and observe collaboration in a different context between working mathematicians on topics of current research interest. One of my motivations was to observe and participate in collaboration in situations where participants possessed various levels of expertise. Even though we expect the primary researchers to have more knowledge and insight than the younger members of the group, there is also the expectation that all members of the group will be able to constructively contribute to the ongoing research. In my own classes at LCC, collaboration has been largely about students learning and digesting key course concepts, but I have also used small group work to facilitate deeper exploration into ideas already presented in class. Sometimes in such group work, one or two students may have insights that can be profitably shared with the whole group and even the entire class. I was interested in whether any insights I gained in this sabbatical experience could inform further efforts into encouraging students to apply knowledge already acquired to obtain a deeper understanding of key concepts.

Sabbatical process

My original plan was to work closely with members of the number theory group at the University of Oregon Mathematics Department, attend seminars locally and at other institutions, and then to accompany members of this research group to the Arizona Winter School in Tucson in March. The first two parts of the plan were successfully carried out in spite of the pandemic through Zoom sessions and email communications. The Winter School program was postponed until March of 2022 because of the pandemic, but the Arizona Winter School did create a “Virtual School in Number Theory” to fill the need to educate young students on current research topics, and I was able to participate in a number of these courses and activities. The Virtual School consisted of four mini-courses, each led by a different instructor, together with problem assignments and discussion groups. Because of the unanticipated demand in the number of applicants and the priority given to young mathematicians just starting out in their professional careers, I was not assigned to a formal discussion group, but I was able to organize an informal group of other similarly interested participants to discuss the lecture material and work together on the homework problems.

Results and outcomes

The exact topics which became central to my work differed somewhat from my original proposal to align more closely with the interests of the University of Oregon number theory group. The central topic was that of cyclotomic fields and *p*-adic L-functions. My main resource for this was "Introduction to Cyclotomic Fields" by Lawrence Washington along with some of the material presented in the Arizona Virtual School. I supplemented the Washington text with other sources to help me review some of the field theory that I had studied years ago and to fill in a few gaps in my knowledge. A bibliography at the end lists all of the main sources I consulted on the sabbatical. I also started studying the topic of modular forms, which was also covered in one of the Virtual School courses, and have (barely) begun introducing myself to the topic of automorphic forms. Professor Ellen Eischen at the University of Oregon and her graduate student Jon Aycock very graciously helped me start finding my way around these topics, and I discovered (as expected) that I was more frequently acting in the role of a student than a teacher in many of our interactions.

Very briefly, fields are sets of numbers with well-behaved operations of addition, subtraction, multiplication, and division. Fields that we work with in our basic algebra classes at LCC are the field of rational numbers (fractions), the field of real numbers, and the field of complex numbers. The set of integers (positive and negative whole numbers) is not a field because division of one whole number by another does not always yield a whole number. We sometimes extend a field by adding in an element not in the original field, such as a square-root or cube-root. Much of the work we do in simplifying radicals (roots) in Math 95 can implicitly be considered as taking place in field extensions of the rational numbers. The study of field extensions has come to be known as Galois theory, named after an early 19^{th} century mathematician who studied the topic of what polynomial equations could be solved entirely by the four basic operations plus the use of radicals. The key to his results was the "Galois group", a group of transformations of the extension field which preserved the basic operations and left the smaller field unchanged. An extension by adding a radical led to Galois groups which were "abelian", named after Galois' contemporary Abel, which meant that two transformations could be performed in either order with the same overall result. Abelian groups are also sometimes called commutative groups.

A cyclotomic field is a field obtained by adding an n^{th} root of 1 to a given field, usually the field of rational numbers. Because the complex number i is a 4^{th} root of 1, for example, we can consider the field extension of the rational numbers obtained by also including i and all numbers which result from the four basic operations. Cyclotomic fields are generally contained in the complex numbers and have many interesting algebraic properties. An important theorem proves that all abelian extensions of the rational numbers must be contained in some cyclotomic field.

One of the key tools in studying cyclotomic fields is the use of "*p*-adic number fields". These are based on an extended concept of absolute value. Our conventional definition of absolute value is what allows us to extend rational numbers to real numbers. Most real numbers, such as pi or the square-root of 2, cannot be expressed as rational numbers, but they can be approximated as closely as desired by rational numbers. By "approximation", we mean that the absolute value of the difference between the actual value and the approximate value can be made as small as desired. A p-adic number field exists for any prime number p, and a p-adic absolute value is considered small if two numbers differ by a large power of the prime p. For example, if $p = 5$, we consider the numbers 2 and 127 to be "close" because $127 – 2 = 125$ which is 5^3, a fairly large power of 5. Such a concept may seem a little counter-intuitive, but it turns out to be very fruitful in analyzing the properties of more conventional number fields, including cyclotomic fields.

In 1900, the mathematician David Hilbert published a list of 23 unsolved problems that he believed would define the direction of mathematical research during the 20th century. The majority of these problems have been solved, but a handful remain. One of the remaining problems is his 12th problem, which is to characterize all extensions of number fields with abelian Galois groups. In March of 2021, significant progress on this problem was announced which claimed to completely solve the problem for number fields which are "totally real", i.e., contained in the real numbers but not extending into the complex numbers. Professor Eischen shared with me the preprint of this paper, and excitedly pointed out that the result depended in an essential way on p-adic analysis. I have just begun studying this result, and hope to continue the collaboration as I piece together different aspects of it.

Finally, other problems which I studied during the course of this sabbatical were addressed by a course at the Arizona virtual school: Given any positive number n, how many solutions are there of the equation $x^2 + y^2 = n$ where x and y are integers? Also, how many solutions are there of the equation $x^2 + y^2 + z^2 + w^2 = n$ in integers? Known as the two-squares and four-squares problems, these problems are intimately connected to the theory of number fields but also lead in a very natural and compelling way to the study of modular functions and modular forms, another exciting topic on the frontier of number theory research. In 1995, a connection between modular forms and *elliptic curves* was proven, which as a corollary proved the most famous previously unproven theorem in number theory, Fermat's Last Theorem, that the equation $x^n + y^n = z^n$ has no solutions in positive integers for n greater than 2. My work in this sabbatical is giving me new insight into this key historical result.

Reflections on the significance of this sabbatical

To be a good teacher, one must also be a good student! This experience was extremely valuable to me as a teacher because I put myself in a situation where I was more frequently functioning as a student than a teacher. However, in a larger context, the distinction between teachers and students is blurry, as we all have insights at times that may even illuminate those who are teachers in the traditional view. I certainly have had the experience as a teacher when an unexpected penetrating question from a student allowed me new insights into a topic I thought I already understood. This sabbatical experience was really fun for me, as some of my most valuable teachers were actually advanced graduate students, and the shared insights and collaboration seemed to be valuable for all participants.

As a teacher, I have had situations in the past where I assigned group work which was part of the basis for assigned grades, where I was concerned that the stronger students essentially gave the weaker students a "free ride". Now I am reassessing this and considering ways in which I can create the space that all students are encouraged to engage at whatever level is appropriate for them. I am considering doing part of the assessment in such activities by asking students to self-assess what they gained from the assignment. This is relevant to all education, not just in math and science, and I would like to encourage teachers in other areas to take this into account if they are not already doing so.

Finally, this sabbatical was relevant specifically to mathematics education, especially the parts of it concerning number fields. Although we do not use this language of abstract algebra in our algebra classes at LCC, the concept of number fields underlies much of the content of our Intermediate and College Algebra classes. I intend to share this with other Mathematics Department instructors in the coming years through colloquia and informal conversation.

Bibliography

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